The Chow ring of hyperkähler manifolds

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It is usually very large: if X has a nontrivial holomorphic form, $CH_0(X)$ cannot be parametrized by an algebraic variety (Roitman).

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(Intuitive reason: by Riemann-Roch, $\dim |C| = g(C)$.)

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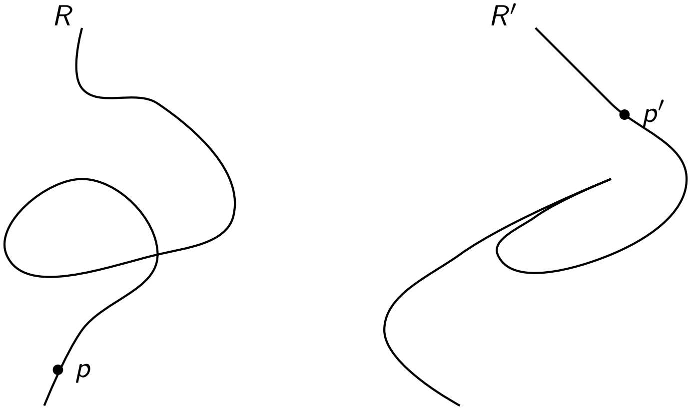

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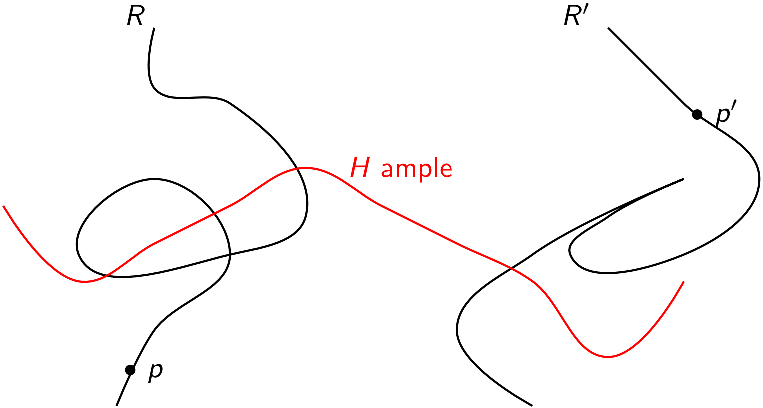

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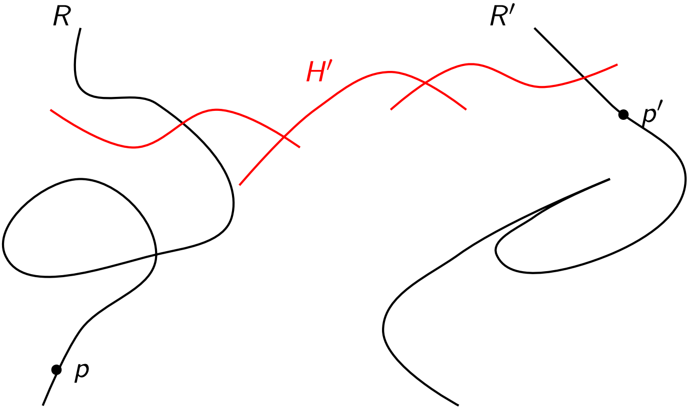

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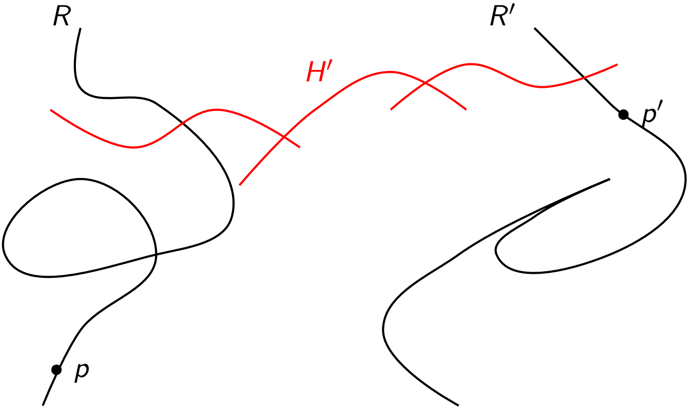

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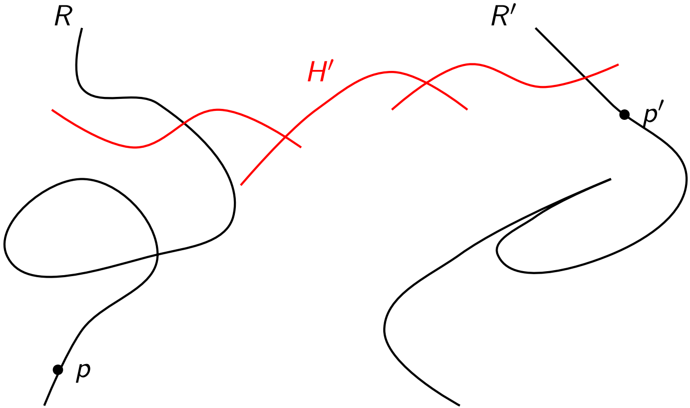

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Proof of ② : $C \cdot C' \sim \sum C \cdot R_i \sim \sum x_{ij}$ with $x_{ij} \in R_i$.

Remarks

③ is much more involved. We deduce it from the vanishing of the *modified diagonal cycle* in $CH_2(S \times S \times S)$ (choosing some $r \in R$):

$$\{(x,x,x)\} - (\{(r,x,x)\} + \text{permutations}) + (\{(r,r,x)\} + \text{permutations})$$

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Remarks

- The vanishing of the modified diagonal cycle has been studied recently by O'Grady, Voisin, Moonen-Yin, with interesting results and conjectures.
- Theorem 1 is quite particular to K3 surfaces: O'Grady has examples of $S_d \subset \mathbb{P}^3$ with $\mathrm{rk}(\mathrm{Im}\,\mu) \geqslant [\frac{d-1}{3}]$.

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Question : For which other varieties do we have such a splitting?

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$CH(A)_{(0)}$ is the space of "symmetrically distinguished cycles". The construction is quite involved (80 pages).

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Voisin has refined (WSP) to incorporate part ③ of Theorem 1 :

(WSP$^+$) The cycle class map is injective on the subalgebra of $CH(X)$ spanned by divisors and the Chern classes of X.

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Question : Is there a larger (natural) class of Calabi-Yau manifolds for which (1) holds?

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Conjecture

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+ 2 sporadic examples in dimension 6 and 10 (O'Grady).

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Recall : For each g, one 19-dimensional moduli space $\mathcal{F}_g$ of K3 surfaces $S \subset \mathbb{P}^g$ ($S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$, etc.)

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Other examples: O'Grady, Iliev-Ranestad, Debarre-Voisin ($n = 2$); Iliev-Kapustka^2-Ranestad ($n = 3$), Lehn^2-Sorger-v. Straten ($n = 4$).

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Then write down complete list of relations between these generators of $DDCH(S^n)$, and check that they hold already in $CH(S^n)$. ∎

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Remark (Q. Yin): Can we go one step further, namely prove $DDCH(S^n) \hookrightarrow H(S^n)$ for $n = 2\tau + 2$?

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So (WSP) for $S^{[n]}$ implies nothing for type $K3^{[n]}$.

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$p^*H^{n+1} = 0$ in $CH(X') \Rightarrow D^{n+1} = \varphi^*p^*H^{n+1} = 0$ in $CH(X)$. ∎

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Hope : For hyperkähler manifolds, the B-B filtration admits a **multiplicative splitting**, i.e. comes from a graded ring structure:

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Recent work of Voisin gives some evidence in the case of CH_0 :

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For $x \in X$, put $\qquad O_x := \{y \in X \mid y \sim_{rat} x\}$.

O_x is a countable union of closed subvarieties Z which are *isotropic* – i.e. $\sigma_{|Z} = 0$. In particular $\dim O_x \leqslant n$.

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Conjecture (Voisin): The filtration $F^\bullet$ and $S^\bullet$ are opposite; i.e., if $CH_{(j)} := S^{n-j} \cap F^{2j}$:

$$CH_0(X) = \overbrace{CH_{(0)} \oplus \ldots \oplus CH_{(2i)}}^{S^{n-i}} \oplus \ldots \oplus \underbrace{CH_{(2j)} \oplus \ldots \oplus CH_{(2n)}}_{F^{2j}} \ .$$

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By expected properties of B-B filtration,

$\Rightarrow\; S^{n-i}CH_0(X) \twoheadrightarrow CH_0(X)/F^{2i+2}$. ∎

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Happy birthday, Ron!